The Serre construction

X sm. projective surface, E rank 2 bundle on X,
$\det(E) = L$.

Suppose $\exists\ s \in H^0(E)$ s.t. s vanishes on a finite set of pts, $Z \subset X$.

(Dual)
Koszul complex is

$$
\begin{tikzcd}
0 \arrow[r] & \mathcal{O} \arrow[r, "s"] & E \arrow[r] & \Lambda^2 E \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & 0 \\
 & & & L &
\end{tikzcd}
$$

exact except here

s vanishes along $Z \Rightarrow \operatorname{coker}(E \to L) = L|_Z$

$\Rightarrow \quad 0 \to \mathcal{O} \to E \to L \otimes I_Z \to 0$ is exact.

<u>Q</u>: Starting instead w/ L and Z, can we construct rk 2 v.b. E s.t. $\exists\ s \in H^0(E)$ vanishing at Z and $\det(E) = L$?

Torsion free!

<u>Idea</u>: Consider $\operatorname{Ext}^1(L \otimes I_Z, \mathcal{O}_X) \ni e \mapsto \quad 0 \to \mathcal{O}_X \to \mathcal{F}_e \to L \otimes I_Z \to 0$

<u>Problem</u>: $\mathcal{F}_e$ may not be a v.b.

Which extensions determine a v.b.?

<u>Note</u>: If $Z' \subseteq Z$, then $L \otimes I_Z \subseteq L \otimes I_{Z'}$.

Thus, starting w/

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O} \arrow[r] & \mathcal{F}' \arrow[r] & L\otimes I_{Z'} \arrow[r] & 0 \\ 0 \arrow[r] & \mathcal{O} \arrow[r] \arrow[u, "id"] & \mathcal{F} \arrow[r] \arrow[u] & L\otimes I_{Z} \arrow[r] \arrow[u, phantom, "\cup\mid"] & 0 \end{tikzcd}$$

pull-back to get

quotient supp. on finite set

so get natural map

$$\beta : \mathrm{Ext}^1(L\otimes I_{Z'}, \mathcal{O}) \longrightarrow \mathrm{Ext}^1(L\otimes I_Z, \mathcal{O})$$

<u>Prop</u>: Given $e \in \mathrm{Ext}^1(L\otimes I_Z, \mathcal{O}_X)$, $\mathcal{F}_e$ is <u>not</u> a vector bundle $\iff \exists\ Z' \subsetneq Z$ s.t. $e \in \mathrm{Im}(\beta)$

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<u>Proof</u>:

Let $e \in \mathrm{Ext}^1(I_Z\otimes L, \mathcal{O})$ and assume $\mathcal{F} = \mathcal{F}_e$ is not a vector bundle.

Idea: $\mathcal{F}^{**}$ is a reflexive sheaf. On smooth varieties, reflexive sheaves are locally free off of a codim ≥ 3 locus.

$\Rightarrow \mathcal{F}^{**}$ is locally free and natural map

$\mu : \mathcal{F} \to \mathcal{F}^{**}$ is an isomorphism away from the locus where $\mathcal{F}$ is not locally free

get exact diagram:

$$\begin{tikzcd} & & 0 \arrow[d] & & \\ 0 \arrow[r] & \mathcal{O}_X \arrow[r] \arrow[d, equal] & \mathcal{F} \arrow[r] \arrow[d] & I_Z\otimes L \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathcal{O}_X \arrow[r] & \mathcal{F}^{**} \arrow[r] \arrow[d] & I_{Z'}\otimes L \arrow[r] \arrow[d] & 0 \\ & & \tau \arrow[r, equal] \arrow[d] & \tau \arrow[d] & \\ & & 0 & 0 & \end{tikzcd}$$

Since $\tau \neq 0$, $Z' \subsetneq Z$, as desired.

$\Leftarrow$ Now assume $\exists\, Z' \subsetneq Z$ s.t. $\mathcal{F} = \mathcal{F}_e$ is the image of $e' \in \mathrm{Ext}^1(I_Z \otimes L)$

Let $\mathcal{E} = \mathcal{F}_{e'}^{**}$. Then get

$$\mathcal{F} \hookrightarrow \mathcal{F}_{e'} \hookrightarrow \mathcal{E}$$

- $\mathcal{F} \hookrightarrow \mathcal{F}_{e'}$: isom. away from finite nonempty set
- $\mathcal{F}_{e'} \hookrightarrow \mathcal{E}$: isom. away from finite (possibly empty set)
- $\mathcal{E}$: locally free, rk 2

$\Rightarrow$ $\mathcal{F} \hookrightarrow \mathcal{E}$ is an isom. except in codim 2 $\Rightarrow$ since $\mathcal{E}$ is a v.b., $\mathcal{F}$ is <u>not</u> a v.b (def)

<u>Cor</u>: $\exists$ $e \in \mathrm{Ext}^1(L \otimes I_Z, \mathcal{O}_X)$ w/ $\mathcal{F}_e$ locally free $\Longleftrightarrow$ $\forall$ $Z' \subsetneq Z$, $\mathrm{Im}(\beta) \subseteq \mathrm{Ext}^1(L \otimes I_Z, \mathcal{O}_X)$ is a proper subset.

<u>Serre-Grothendieck Duality</u>: (See Thm III 7.6 in Hartshorne)
X a sm. projective scheme of dimension n. $\mathcal{G}$ coherent on X.
Then for all $0 \le i \le n$, there are natural isomorphisms

$$\mathrm{Ext}^i(\mathcal{G}, \omega_X) \xrightarrow{\cong} H^{n-i}(\mathcal{G})^*.$$

In particular, in our case, if X is a surface,

$$\mathrm{Ext}^1(\mathcal{G}, \omega_X) \xrightarrow{\cong} H^1(\mathcal{G})^*$$

Proof: see Hartshorne.

Description of map in dim 2 case:

$e \in \mathrm{Ext}^1(\mathcal{G}, \omega_X)$ determines $0 \to \omega_X \to \mathcal{F}_e \to \mathcal{G} \to 0$

$$\Rightarrow \text{induces } f_e : H^1(\mathcal{G}) \to H^2(\omega_X) \underset{\substack{\uparrow \\ \text{fixed}}}{=} \mathbb{C} \Rightarrow f_e \in H^1(\mathcal{G})^*$$

Thm: (Griffiths, Harris) X a sm. surface; $Z \subseteq X$ finite, reduced. Then there exists a rank 2 v.b. E on X with $\det E = L$ and $s \in H^0(E)$ s.t. $Z(s) = Z$

$\Longleftrightarrow$ every section of $\mathcal{O}(K_c + L)$ that vanishes at all but one point of Z also vanishes at the remaining point.

(i.e. Z satisfies Cayley-Bacharach property w.r.t $\mathcal{O}(K_X + L)$

Pf: By corollary, first statement holds $\Longleftrightarrow$ for $Z' \subsetneq Z$

$$\mathrm{Ext}^1(L \otimes I_{Z'}, \mathcal{O}) \to \mathrm{Ext}^1(L \otimes I_Z, \mathcal{O}) \quad \text{is not surjective.}$$

In fact, can check non-surjectivity for $Z' = Z - \{x\}$ $\forall$ $x \in Z$.

i.e. $\exists$ such an $E \Leftrightarrow \mathrm{Ext}^1(L \otimes I_{Z - \{x\}}, \mathcal{O}_X) \to \mathrm{Ext}^1(L \otimes I_Z, \mathcal{O}_X)$ $(*)$ is not surjective. Tensoring both inputs by ω_X gives

$$\mathrm{Ext}^1(L\otimes I_Z, \mathcal{O}_X) \cong \mathrm{Ext}^1(\mathcal{O}_X(K+L)\otimes I_Z, \omega_X) \underset{\substack{\uparrow \\ \text{Serre duality}}}{\cong} H^1(\mathcal{O}_X(K+L)\otimes I_Z)^*$$

and similarly, $\mathrm{Ext}^1(L\otimes I_{Z-\{x\}}, \mathcal{O}_X) \cong H^1(\mathcal{O}_X(K_X+L)\otimes I_{Z-\{x\}})^*$

So non-surjectivity of $(*)$ is equivalent to non-injectivity of

$$H^1(\mathcal{O}_X(K_X+L)\otimes I_Z) \longrightarrow H^1(\mathcal{O}(K_X+L)\otimes I_{Z-\{x\}})$$

This map comes from SES $0 \to I_Z \to I_{Z-\{x\}} \to \mathcal{O}_x \to 0$ twisted by $K+L$.

$\Rightarrow$ non-injectivity is equivalent to $\underbrace{H^0(\mathcal{O}(K+L)\otimes I_{Z-\{x\}})}_{\substack{\| \\ \text{sections of } \mathcal{O}(K+L) \\ \text{vanishing at } Z-\{x\}}} \to \overset{\substack{\mathbb{C} \\ \|}}{H^0(\mathcal{O}_x)}$ not being surjective (i.e. being zero)

This means precisely that every section of $\mathcal{O}(K+L)$ vanishing on $Z-\{x\}$ also vanishes at x. $\square$

<u>Exercise:</u> X sm. surface, C_1, C_2 effective, reduced divisors meeting transversely. Show that $C_1\cap C_2$ satisfies the Cayley–Bacharach Condition with respect to $|K+C_1+C_2|$.

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<u>Exercise:</u> Let $Z = \mathbb{P}^n$ be m distinct points satisfying C-B w.r.t. $|\mathcal{O}(r)|$ (some $r\geq 1$)

(a.) Show $m \geq r+2$.

(b.) If $m\leq 2r+1$, show Z lies on a line.